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Exploring pre-service teachers' knowledge of efficient calculation strategies

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Background: National and international assessments show South Africa's underperformance in mathematics. As many learners already fall behind in the early grades, where foundational number sense should be established, addressing the challenges of number sense in the Foundation Phase (FP) is important. Research recognises the need to better develop pre-service primary teachers' (PSTs') mathematical content knowledge (CK) and pedagogical content knowledge (PCK). Understanding the importance that CK PSTs bring to their tertiary studies is needed.

Aim: The study explores, what knowledge do third-year FP PSTs have of additive reasoning mental mathematics strategies?

Setting: We share data from a larger study of PSTs' CK and PCK of additive reasoning strategies (a key aspect of number sense) before additive reasoning lectures with 80 third-year FP PSTs during their mathematics methodology course at a private institution.

Methods: Participants included 54 of the 59 PSTs who agreed to take part in the study and were present in the first session when the pre-questionnaire (the focus of this article) was administered. Quantitative and qualitative analysis of PSTs' methods used, performance on the pre-questionnaire calculation items and explanations of methods and strategies were carried out.

Results: Results highlight the need for explicit teaching of efficient calculation strategies along with required fluencies for using strategies.

Conclusion: Knowledge of additive reasoning strategies cannot be taken for granted as established at school.

Contribution: The findings of the study highlight the need for explicit teaching of efficient mental strategies such as Bridging Through Ten, Jump Strategy and Rounding and Adjusting in PSTs' studies.

Keywords: primary mathematics; number sense; mental mathematics strategies; pre-service teacher education; content knowledge.

Introduction

Background and rationale

South African underperformance in primary mathematics is partially attributed to an emphasis on either unit-based counting methods and/or standard vertical algorithms without number sense (Graven et al. 2013; Schollar 2008; Weitz & Venkat 2013). The National Mathematics Advisory Panel (NMAP 2008:27) states that number sense, beyond the initial stages of identifying and approximating the magnitude of a small number of objects and counting, requires 'formal instruction' that develops a 'principled understanding of place value, of how whole numbers can be composed and decomposed, and of the meaning of the basic arithmetic operations of addition, subtraction, multiplication, and division'. Number sense (NS) involves being able to work flexibly with numbers through understanding the relationships between numbers (Askew, M., Graven, M., & Venkat, H. 2022; Fosnot & Dolk 2001). Counting is basic to the improvement of NS but should be seen as the first stage in developing it (Naudé & Meier 2014). A well-developed NS is the basic essential needed for all mathematical areas. Explicit instruction, flexibility and effective strategies are required (Bobis et al. 2005; NMAP 2008). Basic number facts that can be rapidly recalled without resorting to inefficient means such as counting are key to developing NS and calculating efficiently (Askew 2009). Learners who can solve problems using concrete

Note: Special Collection: Mental mathematics and number sense in the early grades.

representations need to learn how to apply mental strategies accurately, efficiently and flexibly (Russo & Hopkins 2018). Mastering basic facts such as bonds to 10, or adding 10 to any number, requires multiple opportunities to practice these facts.

South African learners' poor number sense can be linked to limited teacher content knowledge (CK) and pedagogical content knowledge (PCK) (Shulman 1986) of efficient calculation strategies (Venkat & Spaull 2015). In response, and as a result of the authors' experiences of primary pre- and in-service teachers' weak number sense (NS), lecture sessions were implemented at the first author's institution aimed at exposing PSTs to the calculation strategies of Bridging Through Ten (BTT), Jump Strategy (JS) and Rounding and Adjusting (R&A) to improve PSTs' knowledge of efficient mental calculation strategies and knowledge of how to teach these strategies. The lectures drew on the Mental Starters Assessment Programme's (MSAP) materials. The MSAP is currently being rolled out in Grade 3 by the Department of Basic Education (DBE),[1] specifically to address inefficient calculation strategies and an absence of a structural understanding of numbers required for number sense. Because this programme has shown positive results (see Askew et al. 2022; Graven & Venkat 2021), the first author implemented some of the strategies with her group of 80 third-year PSTs in their 'Teaching Foundation Phase Mathematics' course at a private education institution in the Eastern Cape in 2022. This group provided the empirical field for the study. 59 of the 80 PSTs in the third-year cohort agreed to participate in the study; however, only 54 of these PSTs were present in the first session when the pre-questionnaire was administered.

This article focusses on the PSTs' knowledge of additive reasoning strategies prior to the commencement of the lectures that used the MSAP materials. In order to understand the knowledge of additive reasoning strategies that the PSTs had the first author administered a questionnaire prior to the lectures. This questionnaire focussed on assessing knowledge of additive reasoning strategies with a particular interest in three connected strategies, namely BTT, JS and R&A. These were of particular interest as they allow for efficient mental calculation of problems that lend themselves to these strategies. The first part of the pre-questionnaires included the four calculations: 36 + 8; 47 + 29; 63–24; 98 + 99 with a box alongside each asking PSTs to state the method they used to answer each of these. These four calculation items were purposefully chosen because they lend themselves well to the strategies of BTT, JS and R&A, which are important additive reasoning strategies that support mental calculation. The Curriculum and Assessment Policy Statement (CAPS) for Foundation Phase (FP) Mathematics states that learners should develop number concepts that help 'learners to learn about properties of numbers and to develop strategies that can make calculations easier' (p. 9) and that mental mathematics has an important role to play (DBE 2011). A second part of the questionnaire asked PSTs to describe each of these strategies and how they would teach them. Thus, the first part focussed on PSTs' CK while the latter included probing their PCK in relation to these strategies.

As students both showed and explained their method of calculation for the four calculation items and described their knowledge of the methods of BTT, JS and R&A, this pre-questionnaire instrument provides rich data to answer the research question:

- *What knowledge do third-year FP PSTs at one institution have of additive reasoning mathematics strategies?*

Sub questions included:

- Do PSTs use the mental strategies of Bridging Through Ten, Jump Strategy or Rounding and Adjusting in problems that lend themselves well to these strategies? If not, what strategies do they use and do they use them effectively?
- Are PSTs able to describe these three strategies? If so, how do they explain how they might teach them?

While the BTT and JS MSAP pre-tests were conducted in subsequent lectures because these are timed and the intention is that they are performed mentally, they do not provide data on the methods PSTs used. Furthermore, not all 54 PSTs who wrote the pre-questionnaire wrote the BTT or JS pre-tests. For these reasons, we do not draw on this data for this article.

Literature review and the South African context

Kilpatrick, Swafford and Findell (2001:122) emphasised the need for fluency and mental strategies where students learn with understanding 'using a variety of mental strategies' to calculate more efficiently. Teaching mental strategies could address some key challenges in South African primary schools as learners continue to underperform in mathematics (Reddy et al. 2022). In 2008, Schollar (2008) highlighted that:

> 79.5% of Grade Five and 60.3% of Grade 7 children still rely on simple unit counting to solve problems to one degree or another, while 38.1% and 11.5% respectively, of them rely exclusively upon this method. (p. 111)

He argued that learners' inability to manipulate numbers or understand place value is 'clearly the single most important cause of poor learner performance in our schools' (p. iii). Graven et al. (2013) and Weitz and Venkat (2013) found in their research in Gauteng and the Eastern Cape that this problem continues. This provided the rationale for the formation of the MSAP (Graven & Venkat 2021).

The South African Mathematics CAPS (DBE 2011:8) states that mathematics should 'develop mental processes that enhance logical and critical thinking, accuracy and problem solving that will contribute in decision making' and allocates a daily 10-min warmup that lends itself well to faster-paced

1.See https://www.education.gov.za/MSAP2022.aspx.

mental mathematics. In the South African Mathematics CAPS, the content area ‘Numbers, Operations and Relationships’ makes up most of the FP curriculum (65% in Grade 1, 60% in Grade 2 and 55% in Grade 3) (DBE 2011:10). Foundation Phase learners must understand the basic operations of addition, subtraction, multiplication and division (DBE 2011). The MSAP provides teaching materials for teaching six mental mathematics strategies in the 10-min warmup to lessons. Other than doubling and halving, the strategies focus on additive reasoning. Each strategy is taught over 2–3 weeks with pre- and post-tests indicating children’s learning gains. The materials focus teachers on using the empty number line and part-part-whole bar models, with the intention that these become mental images to aid learners in solving additive reasoning problems (see DBE 2021).

According to the Trends in International Mathematics and Science Grade 5 2019 Study (TIMSS), South Africa is one of the five lowest performing countries in mathematics in the world. While analysing the performance of South Africa’s Grade 5 learners, Reddy et al. (2022) found that about two-thirds of the learners participating in the 2019 study did not show grade-appropriate mathematical knowledge and skills. The Eastern Cape (where this study took place) is ranked third lowest out of the nine South African provinces. Although some improvement is noticed in South Africa’s performance from the start of participation in TIMSS to the 2019 results, there is concern that coronavirus disease 2019 (COVID-19) has negatively affected the progress of South Africa’s mathematics education system (Soudien, Reddy & Harvey 2022). Manuel (2022) states that:

> South African 10-year-olds in 2021 know less than 9-year-olds before the pandemic. (p. 1)

In terms of the early grades, Spaull and Kotze’s (2015) study showed that many learners are two grades behind in mathematics by the time they reach Grade 4. Research from the two South African Numeracy Chairs at Wits and Rhodes indicates that a particular problem is that learners complete the FP with very little number sense and continue to use inefficient unit-based calculation methods well beyond the early grades and for calculations in which this is inappropriate (e.g. 10 + 10 + 10) transitioning to inappropriate and incorrect use of the vertical algorithm (e.g. 98 + 2 = 910) (Graven et al. 2013; Weitz & Venkat 2013). Like Schollar (2008), they argue that a structural understanding of numbers is essential for developing foundational NS and that developing mental mathematics strategies (as included in the FP CAPS) is useful for moving away from inefficient strategies and for the development of number sense (Graven & Venkat 2021).

Researchers have criticised teachers who teach the formal written algorithms of whole numbers too soon as children often fail to understand or master the written algorithms (Anghileri, Beishuizen & Van Putten 2002). The data shared in this article attest to this being the case for some PSTs. This often leads to ‘frustration, unhappiness and deteriorating attitudes to mathematics’ (Plunkett 1979:3) as it is linked with a sequence of memorising and forgetting procedures and facts that often do not make sense to the students.

A teaching approach that foregrounds efficient calculation strategies and supports learners (whether school learners or PSTs) to use the part-part-whole relationship of numbers (understanding that every ‘whole’ number can be split up into two parts) – which is key to number sense (Van De Walle, Karp & Bay-Williams 2014) – is needed. This corresponds to South Africa’s CAPS that recognises the need to develop learners’ mental mathematics and improve critical thinking, fluency and problem-solving skills (DBE 2011). However, such a teaching approach requires teachers who themselves have good number sense and can use their structural understanding of numbers to select and perform efficient calculation strategies flexibly depending on the calculation. For example, R&A, with JS, is well suited to efficiently calculate 47 + 29 (i.e. 47 + 30 – 1) – unit counting or the vertical algorithm would take much longer. If teachers’ knowledge limits them to unit-based counting, algorithms or calculators for basic calculations like this then it is unlikely that they will be able to successfully teach learners to use a range of strategies effectively. Thus, addressing PSTs’ number sense becomes crucial in the context of the national rollout of the mental strategies that are part of the MSAP being introduced across the country with Grade 3 learners (DBE 2021).

As mentioned, there is growing concern over primary in- and pre-service teachers’ inadequate conceptual and pedagogical knowledge essential for teaching primary mathematics. Venkat and Spaull (2015) when analysing the 2007 South African SACMEQ mathematics teacher data from 401 Grade 6 nationally representative mathematics teachers found that:

> 79% of Grade 6 mathematics teachers showed content knowledge levels below the Grade 6/7 b, and that the few teachers with higher-level content knowledge are highly inequitably distributed. (p. 121)

Such challenges are similarly and increasingly highlighted in pre-service teacher education pointing to the need to adapt mathematics education courses accordingly. For example, Manuel (2022) found Bachelor of Education students ‘assessed at three universities scored a weak 54% for maths content-based tests meant for primary school pupils’ (p. 1). In another recent study by Bowie, Venkat and Askew (2019) also across three universities, similar gaps in pre-service teacher knowledge were found. They however further highlighted the small gains in knowledge as PSTs progress from the first to the final year arguing that this indicates:

> [*A*] need for student teachers to revisit primary school mathematics in a way that provides a deep understanding of key mathematical concepts in order to be better prepared for future teaching careers. (p. 286)

Given that such findings indicate many final-year PSTs are underprepared to teach mathematics in primary schools, Taylor (2021) suggests that:

> [*U*]nless initial teacher education [*ITE*] is reformed at the same time, continuous professional development [*CPD*] becomes a never-ending task of making marginal differences to the shortcomings of each successive cohort of qualified but incompetent teachers emerging from the universities. (p. 1)

The consistent picture across these studies points to the critical need for greater attention to be paid to developing primary teachers' mathematical CK during their pre-service studies and challenging assumptions that one can consider foundational mathematical knowledge as an established starting point for developing PSTs' PCK.

While studies into PSTs' mathematical CK are relatively new, the finding of this study is that South African pre-service teacher education needs improvement. Carnoy and Chisholm (2008) raised concerns about how teachers were being educated to teach, stating that 'the relatively low level of mathematics knowledge that teachers have in all but the highest student [socioeconomic status] schools is somewhat troubling' (p. 33). Taylor (2008) at the same time cautioned that 'in the hands of teachers whose own conceptual frames are not strong, the results are likely to be disastrous where school knowledge is totally submerged in an unorganized confusion of contrived realism' (p. 276). Taylor's view about teachers' weak mathematical CK negatively impacting their ability to support learners' mathematical progression is widely held (see, e.g., Jenßen et al. 2022).

The intervention aspect (lecture sessions dedicated to teaching efficient additive reasoning mental mathematics strategies) of the broader study was an attempt by the first author to pay attention to these needs within her preparation of FP PSTs for teaching mathematics. This said, the broader study and lectures simultaneously focussed on developing PSTs' PCK taking into account Adler and Reed's (2003) argument that while CK is necessary, it is *not sufficient* for teachers. They emphasise the need to simultaneously develop teachers' PCK to assist learners in making sense of mathematics. They further cautioned that teacher education institutions should avoid being 'complicit' in the continued production of the mathematics crisis.

Framing assumptions

This study is framed by socio constructivist perspective on mathematics learning that holds that knowledge is actively constructed through interactions in one's environment (Cobb 2007). Mathematical proficiency is considered to involve developing five intertwined strands of procedural fluency, conceptual understanding, strategic reasoning, adaptive reasoning and a productive disposition (Kilpatrick et al. 2001) – this notion of mathematical proficiency aligns with the CAPS (DBE 2011). Furthermore, in alignment with the framing assumptions of the MSAP, this study considers that focussing on fluency, reasoning and problem solving is particularly useful (Askew 2012; Graven & Venkat 2021). In demonstrating the way in which these aspects of proficiency are interwoven, Askew (2009) exemplifies how solving 36 + 7 with BTT by first adding 4 to get to 40 and then adding 3 to 40 to get 43 is only effective and efficient:

> [*I*]f you are fluent in knowing what to add to a number to make it up to the next multiple of 10 – speedily and confidently … If children have to use their fingers to count-on, the strategy is pointless; they might as well carry on counting-on in ones. (pp. 27–28)

In relation to the development of mathematics teacher knowledge and proficiency in mathematics teaching, this study is framed by Shulman's (1987:8) seven categories of knowledge that mathematics teachers need. These categories include CK, general pedagogical knowledge, curriculum knowledge, pedagogical CK, knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of educational ends, purposes and values.

In Shulman's (1986) terms knowledge of efficient calculation strategies is CK. Ball, Thames and Phelps (2008:389) however distinguish between specialised content knowledge (SCK) and common content knowledge (CCK) noticing that SCK is 'an important subdomain of "pure" content knowledge unique to the work of teaching' and is 'distinct from the *common content knowledge* needed by teachers and nonteachers alike'. We see knowing and using mental mathematics strategies to efficiently solve calculations in everyday life as 'needed by teachers and nonteachers alike'. As the data will show, many of the PSTs in this study do not know these strategies and many South African adults are equally likely to not know or use these strategies.

Thus, the extent to which knowing such strategies is 'common' or commonly used by adults in general could be questioned. However, being part of the curriculum and based on the wide range of research that points to these strategies being particularly useful for developing a robust structural understanding of numbers, these strategies are *essential* CK for teachers. Thus, while it may not be of concern if an adult does not know JS and continues to rely on a written algorithm or a calculator for adding and subtracting two-digit numbers – it would be of concern if a mathematics teacher does not know how to use efficient strategies effectively.

Because Ball et al. (2008) distinguish SCK as knowledge 'unique to teaching' (p. 389) from CCK, we consider our focus in this article to be on PSTs' CCK. This is because efficient and effective use of mental strategies such as JS are not unique to teaching and we consider these as *useful* 'Common Content Knowledge' (Ball et al. 2008) for all but *essential* for teachers. This said, from here onward we use Shulman's (1986, 1987) term of CK as it aptly captures the construct that we are researching.

In addition to CK, teachers should have the PCK needed to teach these strategies. Pedagogical content knowledge (an amalgam of content and pedagogy) was introduced by Shulman (1987) to refer to the knowledge that is 'most likely to distinguish the understanding of the content specialist from that of the pedagogue' (p. 8). He identified PCK as the 'missing paradigm' or 'blind spot' (p. 7) in pre- and in-service teacher education, which he argued needed to focus on developing in-depth understanding of various knowledge bases and expertise. Pedagogical content knowledge requires the CK that is to be taught to be known by the teacher. While the pre-questionnaire gathered data on both teachers CK and PCK, because none of the PSTs were able to provide accurate descriptions of the BTT, JS or R&A, they were unable to explain how they would teach these strategies. Thus, the primary unit of analysis for this article is PSTs' CK of additive reasoning strategies.

Research context

This study connects with a larger set of studies occurring across nine South African teacher education institutions in collaboration with the two South African Numeracy Chairs investigating and implementing a parallel pre-service teacher education aspect to the Grade 3 Mental Starters Assessment Programme (MSAP). The Mental Mathematics – Work in Learning Programme (MM-WIL) began in 2021 with a group of mathematics teacher educators and/or researchers from a range of higher education institutions who had an interest in exploring the use of aspects of the MSAP in their teacher education programmes (see Graven 2024).

While the MSAP is already being implemented with learners across provinces, the MM-WIL is a more recent project aimed at exploring (and researching) ways to support the development of PSTs' mental mathematics strategies with an aim towards strengthened foundational number sense and developing knowledge for teaching these. Mental Mathematics – Work in Learning Programme is led by the second author with fellow Chair Hamsa Venkat and the first author is a participant in this programme. Participation involves meeting twice a year with the Chairs and mathematics teacher educators and/or researchers from other institutions to share experiences and emerging research findings and present work at the Southern African Association for Research in Mathematics Science and Technology Education.

This study takes place at a private higher education institution in the Eastern Cape and includes students in their third year of study towards a Bachelor of Education Degree (Foundation Phase) and are thus considered PSTs. The PSTs are from different backgrounds with diverse home languages such as English, Afrikaans and isiXhosa. Of interest, all 80 PSTs in the group are female and thus all 54 participants in the study are female. It is not unusual for early grade teaching to be dominated by female teachers.

Research methods and design

The research adopts a qualitative and interpretivist framework, using a case study of one group of third-year PSTs at a single institution. Case studies are frequently used in qualitative research to research specific groups of people and for exploring 'bounded systems' (Merriam 2009; Stake 1995). The sample was a purposive convenience sample (Merriam 2009) in that the participants emerged from the group of PSTs that the first author (as a lecturer) purposively chose to implement the MSAP with and was convenient in the sense that the lecturer already had long-standing established relationships with this group. While 59 of the 80 third-year PSTs agreed to participate in the study, only 54 were present in the session in which the pre-questionnaire was administered. Thus, the sample size for this article is 54 PSTs who wrote the pre-questionnaire. We use descriptive statistics and tables to summarise the frequency of various methods used and the performance of the PSTs across the four calculation items. We use thematic analysis of methods used to provide qualitative analysis of the way methods were used in the four calculations and described by PSTs in the space alongside the four items and for the description of the strategies BTT, JS and R&A.

As noticed earlier, this article focusses on a single data-gathering instrument – the pre-questionnaire. The pre-questionnaire was a non-time-restricted questionnaire that included four calculations. Pre-service primary teachers completed the four calculations: 36 + 8; 47 + 29; 63–24 and 98 + 99 and in a box alongside stated the method used. As this was not under test conditions, occasional discussion or looking to see what others were doing occurred. The questions that followed focussed on getting students to describe the three mental mathematics strategies (BTT, JS and R&A) before exposure to the teaching of strategies using MSAP materials. Pre-service primary teachers were also asked how they might teach these strategies. However, as only two PSTs managed to correctly describe the BTT strategy, we only took into consideration the responses of these two PSTs when it came to explaining how they would teach BTT.

While the questionnaires were repeated after two lecture sessions addressing these strategies, these post-assessments are beyond the scope of this article. The broader study is analysing the shifts in performance on these. All ethical protocols were followed, including receiving gatekeeper permissions and written informed consent from the participating PSTs. All names used are pseudonyms.

Ethical considerations

Ethical approval was obtained from the Rhodes University Ethics Committee (application number: 2022-5717-6915). All ethical protocols were followed and permissions were obtained from PSTs and their respective institutions.

Results

Here we share the findings from the pre-questionnaire four calculation items to answer our research questions:

- *What knowledge do third-year FP PSTs at one institution have of additive reasoning mathematics strategies?*
- *Do PSTs use the mental strategies of Bridging Through Ten, Jump Strategy or Rounding and Adjusting in problems that lend themselves well to these strategies? If not, what strategies do they use and do they use them effectively?*
- *Are PSTs able to describe these three strategies? If so, how do they explain how they might teach them?*

We address the findings in relation to these questions in the following sub-sections:

- Pre-service primary teachers' performance results on the four calculations
- Pre-service primary teachers' methods or strategies used on the four calculations
- Error analysis on the four calculations
- Pre-service primary teachers' descriptions of BTT, JS and R&A and how they might teach these.

Pre-service primary teachers' performance results on the four calculations

While our research questions focussed on PSTs' knowledge or use of specific additive reasoning strategies, it is useful to know the extent to which PSTs were able to successfully answer additive reasoning calculation items with whatever method they chose. Table 1 shows the performance results of the 54 PSTs on the four addition and subtraction problems A to D in the pre-questionnaires.

All 54 PSTs answered all four questions while administered individually were not performed under test conditions. As evident in Table 1, most PSTs managed to answer all four questions correctly. The percentages correct across the four questions were 94%, 91%, 80% and 88%, respectively, for A, B, C and D. While no PST got all four incorrect, there is some concern for those PSTs who did not manage to answer all four correctly. Because the items are Grade 3 simple calculations, items not requiring reading or interpretation (other than the meaning of the + and – symbols), one would wish that all PSTs answer all questions accurately (other than the occasional careless slip or copy error) and efficiently. Question C, the subtraction question, was the most problematic with one-fifth of PSTs (11/54 i.e. 20%) failing to get the answer correct. Our analysis of the methods used and the error analysis that follows points to reasons for the poorer performance on the subtraction item. Furthermore, our analysis indicates that few PSTs used BTT, JS or R&A for any of the items and showed a preference for less efficient methods such as unit counting, the vertical algorithm or breaking down (i.e. separating the tens and units to calculate).

Pre-service primary teachers' methods or strategies used on the four calculation items

Solving calculations both efficiently and accurately is important. Reviewing PSTs' methods used to solve calculation items A–D enables analysis of the efficiency (and accuracy) of the methods used. Table 2 shows the different methods that PSTs used in the pre-questionnaire calculation items as indicated by analysis of their written method in conjunction with their written statement about their method. Some PSTs indicated a combination of methods, for example, breaking down and the use of fingers. These statements were made in the right-hand column next to the four calculations A–D (as can be seen in Figure 1).

Table 2 shows that the dominant method across the four calculations is the vertical algorithm (used by over 42% of PSTs across each of the four items). This is followed by the breaking down method (used by over a quarter of PSTs across each of the four items). That 23 PSTs used the vertical algorithm and another 14 used breaking down – one with fingers (i.e. 36 + 8 = 30 + 6 + 8 = 30 + 14) for a simple calculation involving adding a single digit to a two-digit number speaks to challenges in basic number sense. Understanding BTT and simple rapid recall facts such as 36 + 4 = 40; 8 = 4 + 4; 40 + 4 = 44 should enable the first calculation to be solved efficiently and mentally, while also understanding the JS and/or R&A would enable quick solution of the remaining problems. Unit counting (counting with fingers and counting on) or a different method that

TABLE 1: Frequencies correct and/or incorrect for pre-questionnaire calculations (N = 54).

Items	A 36 + 8	B 47+ 29	C 63 – 24	D 98 + 99
54 PST's	PRE	PRE	PRE	PRE
Correct	51	49	43	48
Incorrect	3	5	11	6
Not answered	0	0	0	0

PST, pre-service primary teachers.

TABLE 2: Frequency of methods used in the four pre-questionnaire items (N = 54).

Pre-questionnaire – 54 PSTs	A 36 + 8	B 47 + 29	C 63 – 24	D 98 + 99
Counted on with fingers	9	4	0	4
Counted on	4	0	0	0
Breaking down	13	24	23	18
Breaking down with fingers	1	1	2	0
Vertical algorithm	23	23	27	28
Vertical algorithm with fingers	1	0	1	0
BTT or shown on number line	3	0	0	0
Jump strategy	0	0	0	0
Rounding and adjusting	0	2	0	4
Answer only – no method indicated	0	0	1	0

PST, pre-service primary teachers; BTT, bridging through ten.

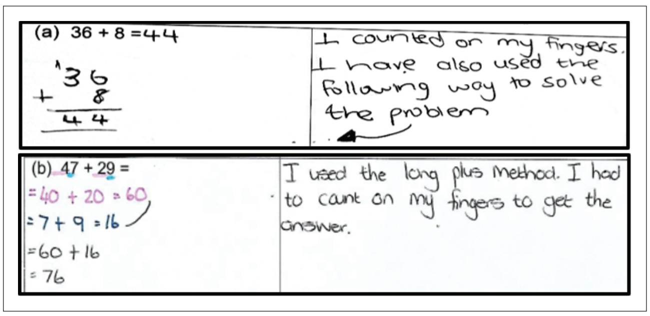

FIGURE 1: Examples of the vertical algorithm and breaking down methods.

included fingers (i.e. those who stated vertical algorithm with fingers or breaking down with fingers) were used by more than a quarter of learners for item A and four PSTs used counting on with fingers for items B and D – although interestingly they did not use this method for the subtraction calculation. While using this method is relatively quick for adding on 8, it is not an efficient method for adding 29 and 99 (as in B and D, respectively).

Only three students used BTT with or without the number line and only for the simplest item A (36 + 8). Similarly, only four students used R&A as a method for the two items that lent themselves well to this method (47 + 29; 98 + 99). Table 2 indicates that the strong performance of the majority of PSTs across the items indicated in Table 1 is predominantly linked to the correct use of inefficient methods of calculation and methods that are dependent on pen and paper. Of interest, no student stated the use of JS although this could have been used in the case of the two students using R&A for 47 + 29 (47 + 30 as 47 jump 30 to 77).

Figure 1 provides examples of the dominant methods of vertical algorithm and the breaking down method (indicated with finger use) used accurately by two PSTs across two items. In the section on error analysis, we discuss the range of errors that PSTs make.

The use of the breaking down method links to its emphasis in the curriculum where it 'is one of the most important techniques in the FP. Using this technique allows learners to split and recombine numbers to help make calculations easier' (DBE 2011:133). The FP CAPS document recommends the introduction of this method from Grade 1 and suggests that learners 'will largely be using this technique in the Intermediate Phase as well' (DBE 2011:219).

For subtraction, the examples given in the CAPS document suggest decomposing in multiples of 10 and counting backwards to make the calculations easier. For example, in Grade 3, the focus of the building up and breaking down method shifts to subtraction of two three-digit numbers with no example of regrouping. In other words, they only suggest 'Breaking down a number into smaller parts to make a calculation easier. Most of the strategies that learners use involve breaking down numbers. They continue to do this with three-digit numbers' (DBE 2011:389). See Westaway (2021) for a summary of the CAPS recommendations for this method for Grades 1–3. While 'breaking down' numbers into tens and units is a method promoted in the curriculum, it is not particularly efficient for these calculations and is highly error-prone especially for subtraction problems in which the units digit of the subtrahend is larger than the units digit of the minuend (as in the case of item C 63-24). A more efficient and likely less error-prone way to solve subtraction calculations would be to use the JS with BTT (63-20 = 43; 43 – 4 = 43 – 3 – 1 = 40 – 1 = 39), which can be performed with the aid of a number line or image of a number line. The Grade 3 CAPS document however suggests the breaking up method with either breaking up both numbers or only the subtrahend. The breaking up both numbers method can work well if each of the *respective* digits in the subtrahend is less than each of the respective digits in the minuend. This is the case in the example given in the CAPS 'clarification notes or teaching guidelines' for Grade 3 Term 2 'Numbers, Operations and Relationships' Topic 1.13 for addition and subtraction (DBE 2011:390):

Breaking up both numbers

$$\begin{aligned} 389 - 137 &= (300 + 80 + 9) - (100 + 30 + 7) \\ &= (300 - 100) + (80 - 30) + (9 - 7) \\ &= 200 + 50 + 2 \\ &= 252 \end{aligned}$$

As Westaway (2021) points out in relation to similar examples given in CAPS for Grade 3 across the terms 1–3, no example involves regrouping that would need learners to subtract with regrouping, for example, for 143 – 87.

The absence of caution in CAPS for using the breaking up method for subtraction irrespective of the numbers involved is cause for concern and likely a contributing factor to the relatively poor results (1 in 5 PSTs got the answer wrong) for our subtraction item C (63-24). Indeed, this method is highly error-prone for subtraction calculations such as our item C. Next, we review the errors PSTs made on the four calculations and analyse these errors.

Error analysis on the four calculation items (N = 54)

Table 1 shows that 3, 5, 11 and 6 PSTs got items A, B, C and D incorrect, respectively. Here we share some of the most common error's made on each of these items.

Figure 2 provides examples of single digit adding errors made by two of the PSTs on Item A (36 + 8). While the

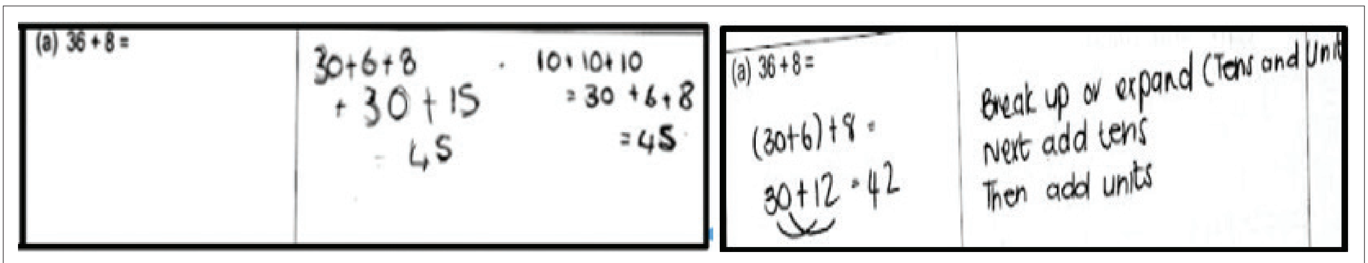

FIGURE 2: Two error examples for item A (36 + 8).

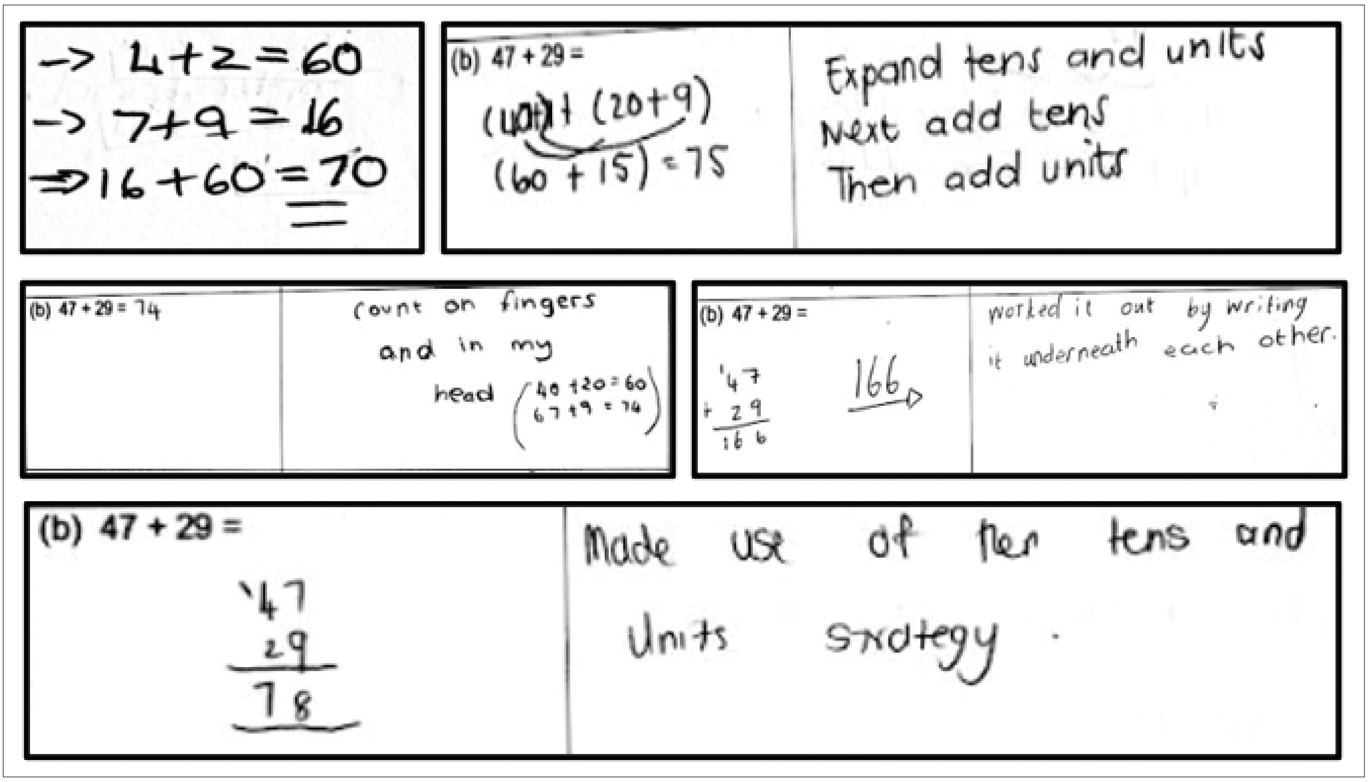

FIGURE 3: Five error examples for item B.

methods used are sound (although not particularly efficient), the absence of fluency in basic addition facts such as 6 + 8 meant these two students answered this incorrectly.

The five errors made by five PSTs on item B (47 + 29) are given in Figure 3. The first two examples show adding errors (adding 16 to a multiple of 10 and a single digit adding errors similar to those above) while building up and breaking down the numbers. The third error involves a PST similarly making an adding error but this time when 'counting on fingers' from 67 (from 47 + 20) to add the 9 to give 74. The vertical algorithm was used incorrectly and without understanding the place value aspect of the algorithm in the fourth example. Thus, while correctly placing the 29 under the 47 and adding the nine and seven units to get 16 (with the 1 as 10 carried to the tens column) when adding the tens, the 1 'ten' carried over is simply transcribed in front of the 6 obtained from adding the two tens' digits. The fifth error indicates the correct application of the vertical algorithm although with a single digit adding error when adding the units 9 and 7 (to make 18) thus giving the incorrect answer of 78.

As shown in Table 1, item C (63-24) was the item where PSTs made the most errors. The discussion above about the CAPS-promoted method of breaking up the numbers while avoiding examples of how to use this method for a problem such as this where the unit digit in the subtrahend (4) is greater than the unit digit in the minuend could partly explain the poor performance on this item. Analysis of the 11 incorrect answers revealed that the most prevalent error involved a reversal of the units (in both the breaking down method and with the vertical algorithm). Seven PSTs reversed the units (shifting the problem from 63-24 to 64-23 to avoid the subtraction of a larger number from a smaller number, thus giving the answer as 41. Figure 4 provides all seven instances of the reversal errors, with the PST in the first instance in the image stating this reversal as her method – 'Again Tens & units. Because we are subtracting, swop the units around 4-3 and subtract the tens'. The reversal of numbers is a common error that arises in subtraction where learners reverse digits to avoid subtracting a larger digit number from a smaller digit. This can be an indication that the learners do not comprehend place value or have a good understanding of base 10 and the number line. This lack of place-value knowledge impacts later learning. This has been seen in other countries at various levels of school, including Senior phase students in the beginning stage of learning formal algebra (e.g. MacGregor & Stacey 1993).

Two of the PSTs who made errors on item C first subtracted the tens but then subtracted the units of both the subtrahend and minuend (60 – 20 – 3 – 4 = 33) as shown in Figure 5.

The last two of the 11 PSTs who made an error on this item seemingly changed the problem (the first ignoring the 3 in 63 and additionally making a single digit subtraction error when working with the units (10 – 4 = 4) while the second made a copy error and instead of subtracting 24 subtracted 29 (see Figure 6).

As evident in Table 2 the most prevalent method for item D (98 + 99) that lends itself well to R&A was the vertical algorithm (28/54) followed by breaking down (17/54)

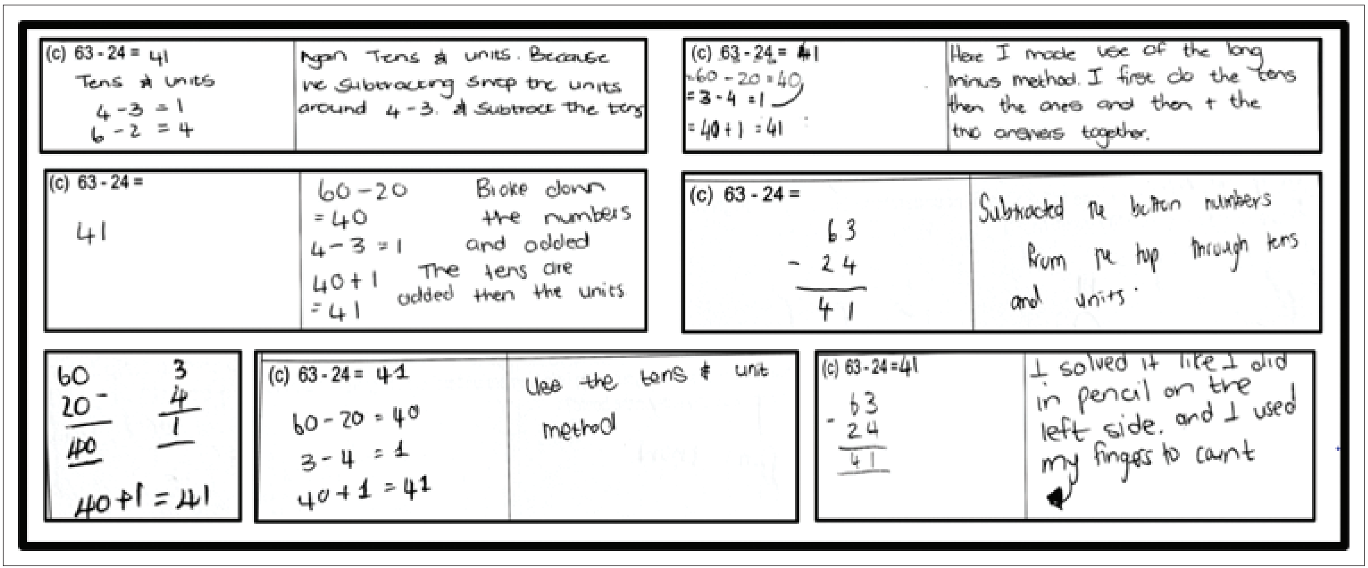

FIGURE 4: Seven reversal error examples for item C.

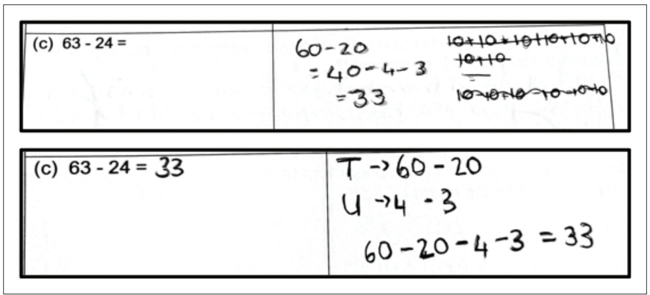

FIGURE 5: Two error examples of breaking down in item C.

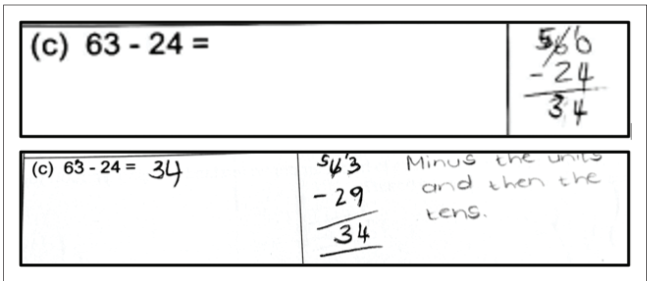

FIGURE 6: Two error examples of transferring the question to the algorithm in item C.

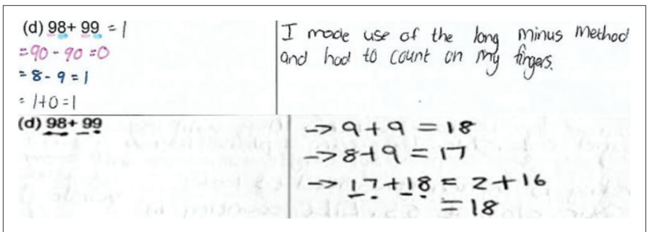

FIGURE 7: Two error examples with the breaking down method in item D.

method – only 4 PSTs used the method of R&A. The six incorrect answers for question D (98 + 99) involved errors while using the former two methods. Of interest, the four PSTs who used R&A all got the item correct. The two entries in Figure 7 show how two PSTs broke down the numbers into units and tens, the first PST then made two

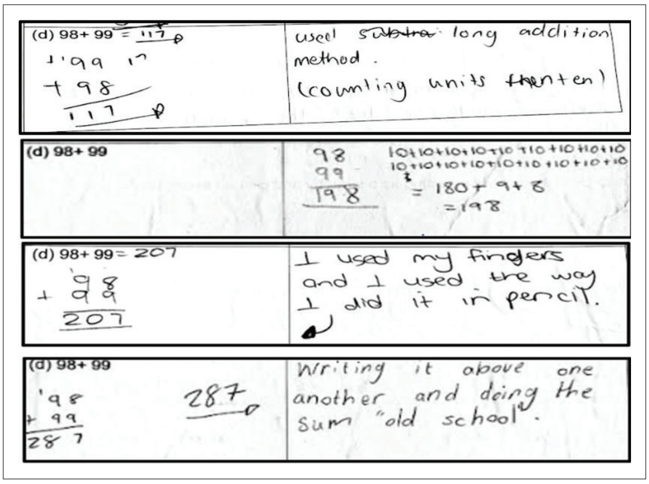

FIGURE 8: Four error examples with the vertical algorithm method for item D.

errors – she subtracted instead of adding (90 – 90 = 0) and answered 8 – 9 as 1 instead of -1. The second while having broken down the tens and units failed to work with the tens as 90 instead simply calculating 9 + 9. She then repeated this error when calculating 18 + 17 answering 2 + 16 (instead of 15) = 18. This indicates a lack of place value understanding and an absence of basic single digit addition facts (i.e. 8 + 7 = 15).

Figure 8 shows the four errors made while using the vertical algorithm for item D. Again, several errors are visible in the single digit calculations performed as part of using the vertical algorithm. In the first example of Figure 8, the PST correctly added the units to 17 and 'carried' the 10 (in 1) but did not seem to know how to do the next stage of the algorithm.

The next two examples show similar errors in working with the addition of the tens and the carried over 10. It is unclear whether these are simply single digit addition errors when totalling the tens. In the third example, the

PST indicated that she counted on with fingers to get an answer of 207, while in the fourth example, the PST added the units, wrote down seven and carried one over – then added the tens together (9 + 9 = 18) and wrote down eight, she then added the one to the one that was carried over earlier to get the answer 287.

Across the examples, it seems that while the four PSTs know some aspect of the algorithm, they do not know how to execute all the steps indicating that they likely do not understand how the algorithm works.

Pre-service primary teachers' descriptions of bridging through ten, jump strategy and rounding and adjusting and how they might teach these

We focus here on the second part of the questionnaire that asked PSTs to describe BTT, JS and R&A and how they might teach these. Analysis of 54 PST responses to these questions indicated that only two PSTs could correctly describe the strategy of BTT while no PST was able to correctly describe JS or R&A. This despite almost half of the PSTs having attempted to describe each strategy with the other half of the PSTs either stating they did not know or did not answer (i.e. they left the space for answering blank). The summary of student responses to the second part of the questionnaire is given in Table 3.

After completing the questionnaire many students commented to the lecturer that they had never heard about or been taught these strategies at school. Examples of PSTs' incorrect descriptions for each strategy are given below:

Description of BTT: Counting the tens and units together.

Description of JS: The jump strategy is that you first count the ones together and if it goes over a ten amount then the ten jumps over to the next ten.

Description of R&A: Rounding is to help the number look more whole so that the number after the comma does not confuse the learner.

As PSTs were unable to describe these strategies they were not in a position to explain how they would teach a strategy they did not know. Thus, the absence of CK in the strategies of BTT, JS and R&A meant that very little data could be gathered on their PCK in relation to these strategies. Post-questionnaire responses could shed light on possible PCK that was developed during lectures while learning about the strategies and how to teach them.

TABLE 3: Frequency of pre-service primary teachers able to describe the strategies bridging through ten, jump strategy and rounding and adjusting (N = 54).

Part 2: pre-questionnaire	Describe BTT	Describe JS	Describe R and A
54 PSTs	PRE	PRE	PRE
Did not know	15	9	5
Answered incorrectly	24	25	31
Left it blank	13	20	18
Answered correctly	2	0	0

PST, pre-service primary teachers; BTT, Bridging Through Ten; JS, Jump Strategy; R&A, Rounding and Adjusting.

For the two PSTs who were able to describe the BTT strategy, they described the strategy quite well paying attention to the way in which getting to the multiple of 10 makes it easier to add. So, for example, the first PST wrote:

> Make numbers nearer to the closest 10s and then add so that it is easier to say 40 + something. E.g. 37 + 6 (take 3 from 6 = 3 to make 37 + 3 = 40 and then add remaining 3.

In answering 'How might you teach this strategy to a Grade 3 learner?' The PST provided a response that indicated some PCK in relation to tools and manipulatives that might support the development of the concept with learners:

> By using blocks or using their number charts. Using blocks allows you to break up whole numbers into pieces and then add.

The use of Unifix blocks could be a useful way to show BTT when adding 37 and 6 as in the PST's example. Of interest, however, is that the empty number line is neither mentioned nor used in the PST's description. The introduction of the use of the empty number line to PSTs for assisting learners using the BTT mentally could thus broaden this PST's repertoire of representations to support teaching this strategy.

The second PST who provided a generally good description of BTT with an appropriate example however stated that for teaching the strategy she would use the counting on technique and draw it (the addend) as you count on. This would indicate that PCK related to the way in which one might teach this strategy is limited (see Figure 9).

Discussion and implications

While most of the PSTs managed to get all four calculation items correct only a few used efficient methods of calculating such as BTT, JS or R&A. The analysis of these results points to the need to expand PSTs' repertoires of additive reasoning calculation strategies so that they are able to flexibly solve calculations effectively and efficiently. Depending on the calculation, they should be able to select the most efficient strategy from a range of strategies as is required in the CAPS (DBE 2011). This is important not only for strengthening their own number sense but also for strengthening their ability to manage multiple-learner methods of calculating and to be able to teach these strategies. Furthermore, the data on PSTs' descriptions of the BTT, JS and R&A strategies indicated that only 2 of the 54 PSTs were able to provide an appropriate explanation for the BTT strategy indicating that most students did not know what these strategies were. This to an extent

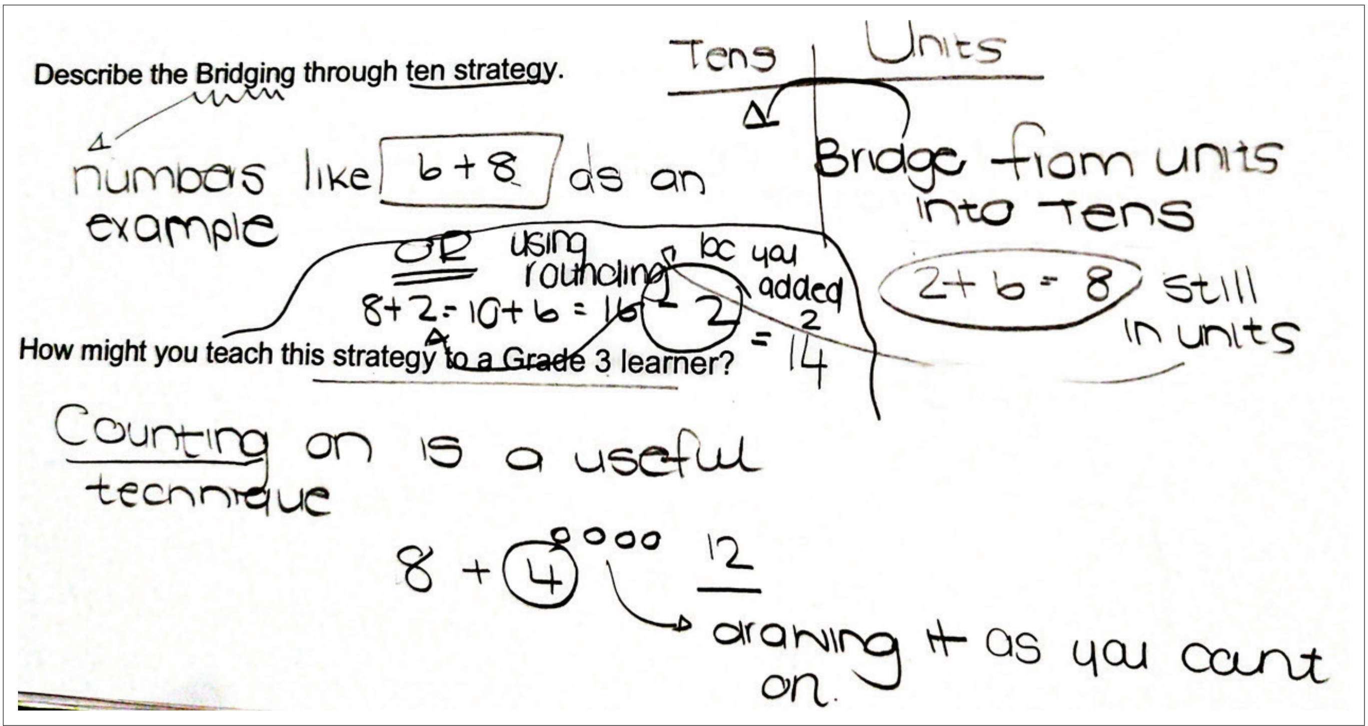

FIGURE 9: Pre-service primary teacher's response to describing the bridging through ten strategies and how to teach it.

explains the absence of the use of these strategies to solve the four calculations – although knowing these strategies is of course not a guarantee for use as people often resort to strategies they feel most confident with (whether that confidence is justified or not).

The data however also point to a group of PSTs who do not manage the four calculations on the questionnaire with any method. That is, for the calculations 36 + 8; 47 + 29; 63 – 24, and 98 + 99, 3, 5, 11 and 6 PSTs answered these incorrectly, respectively, even when they used inefficient methods of counting in ones, the vertical algorithm or the breaking down method. The subtraction item had the highest frequency of incorrect answers partly because of the use of the breaking up method (promoted in the curriculum) that does not lend itself to subtraction items like this one where subtraction of the units digits results in negative numbers (i.e. 3–4 = –1). We were interested to see that several PSTs chose to resolve this problem by simply reversing the digits to answer 4–3. We have highlighted in the article that the CAPS curriculum only provides examples of using the breaking-up method for subtraction problems that do not have this challenge, thus contributing to the problem by not addressing the limitations of that method for certain subtraction calculations.

For this smaller group of students who did not manage to correctly answer the four calculations, irrespective of method, we argue that it is likely insufficient to simply introduce them to a range of efficient calculation strategies as included in the MSAP. Instead, we argue that these PSTs will need targeted interventions that support them in developing both:

- The range of basic facts required for performing calculations across various strategies (e.g. knowing instantly 5 + 3; 26 + 4; 30 + 6; 23 + __ = 30; 46 + 10; 40 + 30, etc.). Without these basic facts, all methods or strategies (including the vertical algorithm and breaking down the tens and units) will be prone to errors.
- The effective use of calculation procedures and strategies, including the effective use of the vertical algorithm and the method of breaking up the tens and units.

As noticed, the broader study conducted pre- and post-assessments to gauge the extent to which the MSAP intervention of teaching the BTT and JS assisted in shifting PSTs' use of methods and performance on various calculation items and description of strategies and how they might teach them. Unfortunately, the time allocated to the intervention (only two lecture sessions) was extremely limited because of the COVID-19 pandemic making it unrealistic to expect much change in the post-assessments. While the data write-up in the first author's Master's thesis is still in process, only modest shifts (improvements) were visible in both performance on the calculations across the group and on the repertoire of strategies used. We therefore expect that should we as lecturers wish to substantively improve PSTs' additive reasoning competence and expand their repertoire of strategies, then substantive time will need to be allocated to this endeavour. Furthermore, we would in future include the development of efficient calculation strategies at the start of PST studies so that the use of the strategies can be developed throughout the 4 years of study.

As lecture time in pre-service teacher education is limited the PSTs who performed particularly poorly on the four calculation items would likely need to embark on an additional programme that is beyond what is offered within the course or to other PSTs who already have this basic knowledge. Computerised programmes are a promising possibility as students could be progressively guided through such courses in their own time and at their own pace. Such programmes would allow for instant assessment and feedback as well as provide the lecturer with feedback on each PST's progress. With strengthened mathematical CCK, the PSTs would be in a better position to engage with the required PCK that is taught in their pre-service teacher education programmes and required for teaching mathematics.

Conclusion

A limitation of this study is that it gathered data from only 54 PSTs in one cohort at a single higher education institution. The data therefore cannot be generalised to PSTs in South Africa. However, from the ongoing discussions and engagement in the MM-WIL project, much of these data resonate with findings in other institutions. A further limitation of the focus of this article on the pre-questionnaire is that while we intended to gather data on PSTs' CK and PCK in relation to mental mathematics strategies (prior to lectures focussed on them), the absence of CK of the strategies meant we were unable to gather meaningful data on their PCK. This points to the importance of CK as a necessary but not sufficient condition for the development of PCK.

While our research shows that most PSTs knew the vertical algorithm or breaking down method of calculating two-digit addition and subtraction problems, few PSTs knew or used efficient additive reasoning strategies. We argue that all PSTs would benefit from learning these strategies to support their own mathematical proficiency and for supporting their teaching thereof. On the other hand, the few PSTs who showed major weaknesses in basic facts and performing calculations accurately using any method need urgent attention. Pre-service primary teachers in their third year embark on extensive teaching practicum in which they will be expected to teach mathematics lessons. It is critical that we find ways as teacher educators to support these students in strengthening their own basic number sense – that 'at-homeness' with numbers (Cockcroft 1982) before they embark on such teaching.

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The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

Authors' contributions

M.K. and M.H.G. contributed to the conceptualisation and writing of the article.

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Data availability

The data that support the findings of this study are available on request from the corresponding author, M.H.G.

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