South African Journal of Childhood Education
ISSN: (Online) 2223-7682, (Print) 2223-7674

Mental mathematics knowledge for teaching of 'high gain' pre-service teachers

Authors:
Hamsa Venkat[1]
Corin D. Mathews[1]

Affiliations:
[1]School of Education, Faculty of Humanities, University of the Witwatersrand, Johannesburg, South Africa

Corresponding author:
Corin Mathews,
corin.mathews@wits.ac.za

Dates:
Received: 11 Jan. 2024
Accepted: 23 Aug. 2024
Published: 11 Oct. 2024

How to cite this article:
Venkat, H. & Mathews, C.D., 2024, 'Mental mathematics knowledge for teaching of 'high gain' pre-service teachers', *South African Journal of Childhood Education* 14(1), a1509. https://doi.org/10.4102/sajce.v14i1.1509

Background: Initial teacher education (ITE) research in South Africa shows gaps in pre-service teachers' (PSTs) primary mathematics knowledge.

Aim: We study the mental mathematics understandings and teaching experiences of three PSTs who achieved high gains for learners they taught mental mathematics to using the Mental Starters Assessment Project (MSAP) jump strategy materials.

Setting: The three PSTs, from one urban university, taught the jump strategy to Grade 3 classes in three different Gauteng schools.

Methods: Learner pre- and post-tests around the taught unit provided the basis for categorising the three 'high gain' PSTs. Extended interviews with each PST were then transcribed. Initial grounded analyses of these data were subsequently overlaid with categories drawn from the mathematical knowledge for teaching literature.

Results: All three PSTs indicated relatively strong common content knowledge of jump strategies and connected specialised content knowledge. They also exhibited strong awareness of the MSAP content. They differed in how they saw the relationship between fluency, calculation and equivalence tasks.

Conclusion: The study's findings indicate the need for more explicit attention to the connection between mental maths fluencies and strategic calculation in ITE.

Contribution: The study points to ways in which mental mathematics can be understood and taught for strong learning gains.

Keywords: mental mathematics; jump strategy; mathematical knowledge for teaching; initial teacher education; South Africa.

Introduction

The South African initial teacher education (ITE) ground in primary education has been the site of extensive recent critique for its purported failings in lifting the quality of language and mathematics teaching and learning outcomes in these areas. Taylor (2021), for example, overviewing a range of other studies, concluded by noting:

> [*U*]ntil the university sector begins to pay more than lip service to the development of professional teachers, the school system cannot move out of its current state of chronic underperformance. (p. 11)

Bowie, Venkat and Askew (2019) had previously identified, through quasi-longitudinal assessment administrations across three institutions, that there was limited, if any, improvement in the primary mathematics content knowledge of primary pre-service teachers (PSTs) in these universities.

In this context of significant gaps in primary teachers' mathematical content knowledge, we were part of an ITE study involving several higher education institutions looking at the mental mathematics teaching and learning materials developed by two South African Numeracy Chair teams in collaboration with the Department of Basic Education, which had begun to roll out nationally in 2022. The materials were linked with the Mental Starters Assessment project (MSAP)[1], an intervention that had been studied and trialled in schools at increasing scales since 2016 with positive outcomes (Graven & Venkat 2021) before going into national implementation in Grade 3. These materials are divided into six units, each attending to one of six mental

1.https://www.education.gov.za/MSAP2022.aspx.

Note: Special Collection: Mental mathematics and number sense in the early grades.

mathematics strategies: bridging through 10, jump strategies, doubling and halving, rounding and adjusting, re-ordering and linking addition and subtraction. Each 3-week unit consists of a 5-min pre-test, eight 15-min lesson starters and two learner worksheets, and a 5-min post-test that is similar to the pre-test.

Within all units, the press is for efficient calculation that moves beyond the counting in or on in ones that continue to be widely documented in South Africa (Porteus & Mostert 2022; Spaull et al. 2022). Each mental mathematics strategy involves this press. For example, using a jump strategy on a calculation such as 87–52 involves breaking down the subtrahend (52) into two jumps based on its place value decomposition (50 and 2). The calculation can then be enacted in two steps, which – initially – can be represented as jumps on a number line and later become mental steps: 87 – 50 = 37 and 37 – 2 = 35. The 'jump strategy' approach is much more efficient than counting back in ones from 87.

In each unit, three task types are included to support understanding and use of the focal strategy: fluency tasks, strategic calculation or strategy tasks and equivalence tasks. Fluency tasks focus on the underlying number of facts needed at the level of rapid recall for children to work with the focal strategy. Fluency refers here to what Hopkins and Bayliss (2017:19) describe as: 'the direct retrieval of an answer from a store of facts held in long-term memory'. In the case of jump strategies, underlying fluencies for a strategy-focussed task such as 87 – 33 include: counting back in 10s from 87 (i.e. 87, 77, 67, 57) and subtracting a single-digit number from any given number (57 – 3 in this case). Stepping up this fluency involves moving to subtracting 30 in one step rather than four backward jumps of 10. In strategy tasks where a bridging through 10 step is involved in the units jump (e.g. 83 – 34), fluencies include knowing how to jump to the multiple of 10 that comes after or before any given number. In this case, strategic calculation using the jump strategy would involve fluency by subtracting 30 from 83, to give 53, then subtracting 3 to give 50 and subtracting the remaining 1 to give 49, that is, 83 – 34 = 83 – 30 – 3 – 1 = 49. These fluencies are needed to complete calculations using the jump strategy. Without these fluencies, the recourse is to counting on or back in ones. We also included equivalence tasks such as: 61 – 32 = 61 – __ – 2. In these tasks, the focus is on understanding the structural equivalence that underlies the use of the jump strategy, rather than carrying out the calculation.

Each lesson starter, designed for use in the curriculum-mandated mental and oral section that should begin all mathematics lessons in primary schools, includes:

- A 1 min–2 min warm-up activity on one or other of the fluencies that underpin the focal strategy
- A 10-min whole class section on the focal strategy (usually involving two core examples led and shared by the class teacher that include attention to the equivalence structure of the strategy)
- A short individual work section on similar examples for children to complete on their own.

The pre- and post-tests were each composed of 20 fluency items comprising the fluency task set and 10 strategy and/or equivalence-oriented items. An example of each task type within the jump strategy unit assessment is shown in Figure 1.

In our work in one urban higher education institution, the second author led an intervention with the 3rd-year BEd cohort using the MSAP materials that focussed on developing PSTs' knowledge of, and work with, these materials in classrooms during their practicum periods. We sought, through 2 weeks of lecture and tutorial time, to share and discuss the model, content and rationales for emphasis on mental mathematics in the MSAP materials to develop students' knowledge of these aspects. This was followed by the students leading the teaching of, and assessment around, the Jump Strategies unit during their practicum period in schools that gave permission for this. Part of the teaching was to make the explicit connection between fluencies and strategies. On the research side of this study, we were interested – narrowly on the one hand in the extent to which our PSTs would be able to impact positively on children's mental mathematics learning outcomes through their teaching of this unit, and more broadly, in how the MSAP materials and their rationales were understood and used by our PSTs.

While some PSTs were allocated classes in grades other than Grade 3 for the Jump Strategies teaching, 35 of the 78 3rd-year BEd Foundation Phase students were able to complete the teaching of the Jump Strategies unit with a Grade 3 class in their schools and submitted their pre-and post-test data, with the relevant informed consents for use of their submitted data for research purposes from all parties (child, parent or guardian, school and PSTs as well as from the university). Our analysis of the pre- and post-test data from these 35 students, teaching in a range of schools (government and private, fee-paying and no-fee schools and suburban and township) in Gauteng province, was based on assessment responses for n = 767 learners with matched data. The outcomes indicated a pre-test mean of 40.1% and a post-test mean of 55.1%. This 15-percentage point average

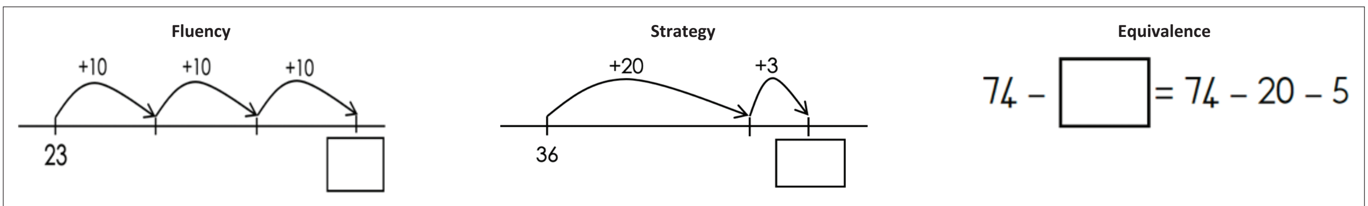

FIGURE 1: Fluency, strategy and equivalence item examples within the jump strategy unit assessment.

improvement compared well with gains reported in the earlier trials of the MSAP materials in which units had been led by in-service, rather than pre-service teachers (Askew, Graven & Venkat 2022).

This was a very promising initial outcome in the context of the critique of ITE that we began this article with. Within the dataset, we noted a further layer that appeared important to probe in trying to understand the potential for positive changes in ITE Foundation Phase mathematics teaching and learning for development. Specifically, there were three students whose classes' performance indicated mean gains of 30 or more percentage points between their pre- and post-assessments. Understanding the experiences of these students, and the inferences we could make about the nature of their mathematical knowledge for teaching (MKT) linked to mental mathematics as presented in the MSAP materials, offered routes into understanding the possibilities for improving teaching and outcomes on the ground within ITE.

Based on data drawn from in-depth interviews conducted with each of these three students on their experiences of learning about the MSAP materials within their BEd course and their experiences of teaching the Jump Strategies unit while on practicum, we focussed on the following guiding research question:

- *What knowledge linked to mental mathematics could be inferred from the reflections of high gain students on their MSAP materials course learning and classroom teaching with the MSAP Jump Strategies materials?*

Our work on the research side of our development work was highly exploratory. While we were well versed in the literature on mental mathematics and its teaching, the problems around coherence that have been written about in South African in-service primary mathematics teaching (Mathews 2021), even when working with relatively structured tasks and materials, as well as the aforementioned concerns around PSTs' mathematical knowledge, meant that we had few predetermined expectations about how the Jump Strategies unit would be used. This led to an initially grounded analysis of our interview transcript data that produced emerging themes, which we subsequently overlaid with some of the categories in Ball, Thames and Phelps's (2008) MKT framework. Our reasons for working in this way are explained in the research methods section.

In this article, we begin with a brief overview of the knowledge base that underpins strong mental mathematics. This provides a useful background to considering the knowledge base required for teaching mental mathematics that we need to be alert to in our work in ITE.

The nature of strong mental mathematics knowledge

Baroody, Torbeyns and Verschaffel (2009) emphasise that mental mathematics underpinned by a strong number sense involves a highly interconnected knowledge base. Describing a trajectory from initial counting-based enumeration, into reasoning strategies based on using known facts and relations to derive further results, and thereon into a gradually expanding base of recalled results in a 'mastery' phase, they note that the 'meaningful memorisation' that underlies an expanding mastery is comprised of 'a rich and well-interconnected web of factual, strategic (procedural), and conceptual knowledge' that produces what they call 'adaptive expertise'. They define adaptive expertise as 'well-understood knowledge that can be applied efficiently, appropriately and flexibly to new, as well as familiar, tasks (mastery with fluency)' (p. 70). The MSAP model of linking fluency tasks with calculation and equivalence tasks rests on this number sense perspective. Given this, an interconnected knowledge base relating to mental mathematics on the part of the teacher is important to develop in ITE as the base upon which good-quality teaching can be built.

Instruction for strong mental mathematics teaching would need to take in an awareness of the trajectory of needing to move on from counting-based approaches to derived and recalled results. This is set within the broader writing noting the importance of awareness of mathematical trajectories at all levels of mathematics instruction, including the early years of schooling (Clements & Sarama 2020). Messages about moving on from counting recur frequently across the MSAP Teacher Guide, but it needs to be noted that flexibly responsive instruction has been noted as limited in the South Africa (Abdulhamid 2017). Shalem et al. (2017) have pointed out that even in the context of scripted lessons:

> '[*K*]nowing and working with learner misunderstandings and scaffolding the complexity of subject matter depends on teachers' knowledge.' (p. 29)

Theoretical framing

The lack of a clear sense of what students' reflections on their learning and teaching experiences could consist of led us to an initially open, grounded analysis of our interview transcript data. We followed Wolcott's (1994) three-frame model of moving from data excerpts to thematic summaries that were interpretively 'close' to the data and then into analytical themes informed by the literature base. It was in the third framing that we noted that several of our analytic themes overlapped with elements of Ball et al.'s (2008) categories of MKT, with the ITE students' reflections on their learning and teaching allowing us to make inferences about the nature of their content and pedagogic content knowledge linked to Jump Strategies and the MSAP materials more generally.

In this section, we detail the key categories of interest within Ball et al. (2008) model and include comments on how the 'general' descriptions of categories offered in this work can be tailored to our focus on mental mathematics. Ball et al. (2008) begin by distinguishing MKT into two key categories: subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Subject matter knowledge, in their formulation, refers to knowledge that is linked directly to mathematics within the work of teaching. They give examples such as deciding whether a mathematical procedure holds generally, or whether a mathematical argument is valid as illustrative of SMK. Pedagogical content knowledge, in contrast, refers to mathematical knowledge linked to teaching, students or curricula. Subject matter knowledge is sub-divided into common content knowledge, specialised content knowledge (SCK) and horizon knowledge, with brief descriptions of what each sub-category consists of:

- Common content knowledge (CCK): Ball et al. (2008) describe CCK in terms of the kinds of mathematical knowledge held by numerate adults including, but not confined to, teachers. This knowledge base includes being able to correctly solve problems, recognising errors and using appropriate mathematical terminology and notation. Being able to work efficiently with Foundation Phase mental calculation strategies as part of basic number sense is commonly described as an important part of adult numeracy and thus falls squarely within the CCK remit. Teachers, of course, need this knowledge, but Ball et al. (2008) persuasively make the case that CCK is necessary, but not sufficient for good-quality mathematics teaching. However, CCK is a category of interest per se in the South African context, given the evidence from primary schooling of teachers not always being able to manage to do – for themselves – mathematics at the level of their teaching (Bowie et al. 2019).
- Specialised content knowledge: Ball et al. (2008) refer to SCK as 'the special mathematical thinking that teachers must do and understand in order to teach Mathematics' (p. 398). This quote points to the SCK category focus on teachers' ways of working with mathematics that are pertinent to teaching but a priori of their working with mathematics in the context of teaching. Included in SCK are: knowing the source of errors; understanding the breadth of applicability of a procedure and the range of contexts where a procedure can be applied; rationales for procedures, choosing, making and using appropriate representations and which alternative approaches work and when. In the context of early mental mathematics, this includes recognising which strategy is most appropriate for particular tasks, for example, being aware that while a jump strategy with bridging through 10 can be used to solve a task like 47 + 19, a compensation strategy involving adding 20 and then subtracting 1 is more efficient. It also involves an awareness of how fluencies and strategies are linked, and how the idea of equivalence is structured into each strategy. This attention to understanding of mathematical connections within SCK was useful in our analysis.
- Horizon knowledge: Ball and Bass (2009:1) describe this aspect as related to: 'a view of the larger mathematical landscape that teaching requires' and Ball et al. (2008:403) describe horizon knowledge as inferable in the 'awareness of how mathematical topics are related over the span of mathematics included in the curriculum'. In the context of early mental mathematics, horizon knowledge can refer to connections between mental mathematics and the rest of the mathematics curriculum as evidence of this kind of broader vision of the mathematical landscape. However, given our specific focus on mental mathematics in this article, this category is backgrounded in our analysis.

Ball et al. (2008) divides the second category of PCK into the following aspects:

- Knowledge of content and students (KCS): Ball et al. (2008) describe this as 'knowledge that combines knowing about students and knowing about mathematics' (p. 401). This category includes features such as anticipating children's difficulties with particular tasks and interpreting children's thinking. In a context where responsive teaching has been noted as limited (Abdulhamid & Venkat 2018), we were interested in PSTs' expressions of awareness of children's thinking and misconceptions and their decisions on how to respond to these.
- Knowledge of content and curriculum (KCC). This category relates to an awareness of the role and place of particular topics within the broader curriculum, including their connections and place in progression hierarchies. In terms of this category, we were interested in the comments that students made about connections between fluencies and strategies within the Jump Strategies unit and the place and importance of mental mathematics in relation to the broader mathematics curriculum.
- Knowledge of content and teaching (KCT): 'combines knowing about teaching and knowing about mathematics' (p. 401). This category includes aspects such as choosing the most appropriate sequences of examples, and the representations most likely to be helpful to assist children to understand ideas and their connections. In relation to this category, we probed details of insertions and adjustments that the students reflected on making to tasks and task sequences and to the representations provided in the MSAP materials.

While at one level, it is entirely reasonable to expect that PST learning related to early mental mathematics should not need to be at the CCK level, South African evidence points to the likelihood of needing to attend to CCK, SCK and the PCK aspects in BEd programmes. In recent writing, Porteus (2023) describes the in-service early-grade teachers she has worked with over an extended period in South Africa as having a 'fragile relationship with mathematics', with some relying themselves on unit counting strategies, rather than leveraging number relationships generally and base 10 relationships in particular for more efficient calculation. We were thus interested in what our high-gain student reflections indicated about both their SMK and their pedagogic content knowledge.

Research methods and design

As noted already, the three high-gain students we were focussed on had all produced a 30+ percentage point gain based on matched learners in the Grade 3 practicum classes they were working with. All three students fell into the top quartile of attainment on their 1st- and 2nd-year mathematics courses. Ellen, Tsolo and Hlonipha (all pseudonyms) completed their third-year practicum in different school types: Ellen was in a suburban private school serving a relatively advantaged, racially diverse population of learners. There were 19 learners in the class and matched pre- and post-test results were submitted for 18 learners. Tsolo's practicum was in a fee-paying government township school serving an entirely black South African and/or African learner population. There were 45 learners in the class, and matched pre- and post-test results were submitted for 39 learners. Hlonipha was in a no-fee government township school serving an entirely black South African and/or African learner population. There were 73 learners in the class and matched pre- and post-test results were submitted for 27 learners – with strike action in the transport sector leading to parents keeping children at home on days with limited transport and potential violence on the streets. We note that the low proportion of matched learners in Hlonipha's school may have affected the size of the gain we have associated with her matched data. The mean pre-test fluency and calculation or equivalence item scores for 32 other children who were present for the pre-test and then absent for the post-test administration were lower than the parallel scores for children in the matched sample. We know that many of these children were in most of the eight lesson starters that Hlonipha taught. Given that the conditions of teaching and learning vary substantially in South Africa between no-fee and more advantaged fee-paying schools in terms of numbers of learners in classrooms, school resources, the extent of individuation and specialisation of focus on mathematics (Hoadley 2007), we decided to keep Hlonipha within the current case study based on two factors: her large gain based on the matched learner sample and the advantages of retaining her experiences of teaching the MSAP unit in a no-fee school within our sample.

The ethics process involved the PSTs, the principal, parents and learners all receiving information letters about the project that noted that there would be no consequences for them if they chose not to participate in the research and submitting written informed consent forms.

Beneath the high mean gains that were produced by the three PSTs' classes from pre- to post-test, there were differences in underlying performance across the 20 fluency items and the 10 calculation or equivalence items – see Table 1 for a summary of the results for each PST based on the mean scores of their matched learners' pre- and post-test results.

What we see in this data is that – reflecting broader South African patterns, Ellen's more affluent learners showed higher pre-test performance on both fluencies and calculations and/or equivalence than the learners in the other two schools. Tsolo's learners showed particularly high mean gains on the fluencies, while Hlonipha's learners (acknowledging the low proportion of matched data) showed the highest level of gains on the calculation and/or equivalence items.

Taken together, these differences in patterns of gains suggested that we had some interesting outcomes to probe in our interviews within this high-gain group. All three students were invited to participate in online interviews where they would be asked to reflect on their experiences of learning about the MSAP materials in lectures and their reflections on teaching the Jump Strategies unit during their last practicum. An important aspect to note is that the interviews were conducted between January and March 2023; some 4–5 months after the end of the focal practicum, but – as the interview excerpts reflect – all three students referred in specific terms to the MSAP materials and their lesson plans and reflections during the interviews. The interviews were between 34 min and 43 min long. The interviews were transcribed and followed by an initial grounded analysis of those transcriptions. Our emerging themes overlapped with some of the categories in Ball et al.'s (2008) MKT model, in particular, CCK and SCK on the SMK side, and combinations of all the PCK categories in relation to the MSAP materials. Our analysis is therefore informed by this framing.

Ethical considerations

Ethical clearance to conduct this study was obtained from the University of the Witwatersrand Human Research Ethics Committee (Non-Medical) (reference no.: H21/11/45).

TABLE 1: Overview of mean scores for correct (✓)/incorrect (x)/blank responses for Ellen, Tsolo and Hlonipha's classes on jump strategies pre- and post-tests.

Pre-test						**Post-test**					
Fluencies/20			**Calculations and equivalence(10)**			**Fluencies(20)**			**Calculations and equivalence(10)**		
Ellen (n = 18 matched learners)											
✓	X	blank	✓	x	blank	✓	x	blank	✓	x	blank
9.5	1.7	8.3	3.2	1.8	5.1	17.9	1.7	0.4	6.1	1.3	2.7
Tsolo (n = 39 matched learners)											
✓	X	blank	✓	x	blank	✓	x	blank	✓	x	blank
6.9	11.3	1.8	0.9	8.3	0.8	16.1	3.8	0.1	2.9	7.0	0.1
Hlonipha (n = 27 matched learners)											
✓	X	blank	✓	x	blank	✓	x	blank	✓	x	blank
6.1	10.6	3.3	0.9	7	2.1	11.8	7.7	0.4	5.1	2.7	2.2

Results and analysis

In this section, we draw on excerpts from the three PSTs' interviews in ways that detail the commonalities and the differences in their experiences and the inferences we could draw about their MKT. Given that several of the PSTs' reflections drew from incidents of their teaching, we do not focus on KCT as a separate category within PCK; instead, we use their descriptions of and reflections on teaching to point to what we could infer about their wider MKT base.

On the SMK side, PSTs' descriptions of their teaching within the interviews included fluent and coherent articulations of using the jump strategy. In their own fluent working with the jump strategy, we could see indications of a strong CCK base. All three stated that they had not heard of the term 'jump strategies' before seeing this unit in the MSAP materials but recognised that it offered an efficient calculation approach, supported by the number line representation. This content-oriented understanding of relative efficiency and linked representation goes beyond a 'lay' understanding of the jump strategy and therefore into the SCK terrain.

However, there were differences between the PSTs in the extent to which they could detail the connections between the fluency and the calculation or equivalence items – which we would represent as part of SCK when the context of this discussion was focussed on the mathematical ideas involved. In the illustrative excerpts that follow, a specialised understanding of mathematics is often communicated within and alongside actions that can be linked with PCK categories. We focus on the SCK aspects first and subsequently deal with PCK-related aspects later in this section.

As noted earlier, the unit and its associated pre- and post-assessments included items focussed on equivalence structure, rather than calculation. Examples of these kinds of questions in the Jump Strategies pre-test were as follows:

> Q9. 61 – 32 = 61 – __ – 2 and Q10. 74 – __ = 74 – 20 – 5

These questions were aimed at developing awareness of the place value decomposition relations that underpin Jump Strategies, described in other writing as working with early numbers algebraically, rather than arithmetically, given the emphasis on number structure (Venkat, Askew & Graven 2023). In over a decade of work in South African earlygrade classrooms, we have very rarely seen these kinds of structure-oriented tasks in use, and Ellen acknowledged her unfamiliarity with such questions:

> 'I'm gonna be honest. It took me a while to figure out what was happening in that question [*referring to 61 – 32 = 61 – __ – 2*] and then when it clicked, I felt so stupid for not realizing it earlier.' (Ellen)

When asked to explain what had clicked, Ellen explained using the example of 61 – 52 = 61 – _ – 2:

> 'I can't even remember how I realized what was going on … it was 61 minus 52 = 61 minus blank minus 2, and then I was like, okay, they're just missing the number. … We have the 61 on the other side, we have the 2 on the other side. So what number are we missing then? They were like, ohh okay, we're missing 50.' (Ellen)

In this response, we could see that some aspects of the work on connecting between equivalence tasks and the core focus on using jump strategies were new to Ellen, but that she was able to make these connections in working with the tasks, seen – as we show later in this section – in her comments on how she worked with tasks like these in her teaching.

In her reflections on her teaching, we noted several instances of Ellen noticing the connections between the fluency tasks and the focal strategy and – in some cases – adapting tasks to strengthen these connections. By way of example, Ellen noted that while the warm-up tasks in starters 1–3 (all focussed in different representations on 10 more or less than a given number) linked well with the focal strategy tasks in the starters, this was not the case for starter 4, where the warm-up was focussed on stating the multiple of 10 after a given number. Subsequently, though, she offered an example and a way of working with it that would require knowing the next multiple of 10 – working through 23 + 12 as 23 + 7 + 5. While this approach did not use the jump strategy's typical place value decomposition of the addend of 12 into 10 and 2, Ellen was then able to note that any example involving a bridging through 10 step would need awareness of the next multiple of 10.

Hlonipha also described her planning process as involving looking at the core problems to be worked with in a starter and then creating tasks that she felt would be useful to offer her learners better grounding in the fluencies required to work with the core strategy. For example, referring to starter 7, where the jump strategy incorporated the need for a bridging through 10 step, she described a lead-in task that she included:

> '[*L*]ike when we had to do lesson seven, which included bridging through ten, I started by writing numbers on the board from 20 to 40. I asked them which are multiples of 10, and they would say 30, 40. Then I will circle those numbers. Then I will say 23, counting forward, which multiple of 10 will I find? Which first multiple of 10 will I call out? Then they would say 30. Then I will say 23, and count backwards. Which multiple of 10 will …? I find they will call out 20.' (Hlonipha)

Hlonipa used this task as a preamble to the rapid recall warm-up activity in starter 7, which simply involved the teacher stating a number and the class calling out the multiple of 10 before this number. Notable for us in Hlonipha's work was the unpacking of this idea using a longer number sequence, with her inscriptional choices of circling the next multiple and previous multiples of 10 providing a contrast between these numbers and other numbers. Hlonipha explained her sense that for the learners in her class: 'I thought just coming to them and saying round up to 10 would not work'. More generally, Hlonipha, more than the other two PSTs, communicated a sense of the need for learners to 'have mastered the rapid recall skills' in order to successfully access and complete the focal strategy tasks in each starter.

Tsolo's responses tended to be less well articulated on the relationships between the fluency and calculation and/or equivalence tasks. When we asked her about her sense of the similarities and differences between the two item sets, she noted features such as: 'they both use the number line and I think it's the same strategy' as similarities and the calculation and/or equivalence tasks as involving 'bigger numbers' and the fluency items being 'easier' as differences. In contrast to Hlonipha, there was no detail on the ways in which strategy task calculations *include* fluencies. Tsolo's primary attention appeared to be on moving children off counting in ones – she noted, for example, that in the whole class teaching focal strategy section that followed the work with fluency tasks, she had focussed on recall of number bonds as another route into avoiding count based.

Given our earlier point about fluencies being necessary for efficient working with calculation strategies, Tsolo's more isolated focus on recalling basic number facts without counting in ones represents a narrower SCK in comparison to Ellen and Hlonipha's sense of fluencies connected with strategies. Ellen, additionally, explained her intentional juxtaposition of tasks 9 and 10 above in her instruction to emphasise their similarity:

> '[*S*]o I basically used the two [*items above*] in relation to each other to explain what they wanted. So like I said, for example in question nine, they have 32 and then I said, OK, we have the 61 on the other side [*pointing to the 61s on either side of the equation*], we have the two on the other side [*pointing to the 2 in 32 and the 2 on the right hand side*]. So what number are we missing then? They were like, ohh, OK, we missing 30.' (Ellen)

> 'And then in question 10, I said OK. So on both sides there needs to be the same thing. So we have the 74 on both sides. Then we have 20 on the side and we have a 5 on the side. And if you put 20 and 5 together, what do you get? Then they got to 25.' (Ellen)

While there is a lack of explicit focus on the subtraction operation in this explanation, in this way of working, Ellen directs learners' attention beyond producing the solution to a particular task by pointing out links and general ideas across them. Ellen did not make explicit the ways in which these tasks differed from calculation tasks, but in her connecting these two tasks and comparing the quantities on either side of the identity rather than working out the missing number 'operationally' (Stephens & Ribeiro 2012) through calculating the answer to one side and then making the other side equal to that answer, she showed some awareness of equivalence as an important idea within early number learning, a feature that has been described as limited in the South African literature (Essien et al. 2023). Ellen's comments therefore indicate an SCK that reflects awareness of how equivalence-oriented problems differ from calculation-oriented problems.

In Ellen's reflections, there were also indications of a co-development of common and SCK, as she indicated that listening to peer presentations of processes of working through individual tasks and task sequences and getting a chance to share the teaching of these tasks in tutorial sessions sometimes surfaced her own misunderstandings or offered clearer explanations for solutions than she had considered:

> '[*I*]t also could happen occasionally where I maybe misunderstood the question, so I answered it how I interpreted it, but then it turns out that everyone else interpreted it in a different way or solved it in a different way that was maybe an easier way, or a different way to what I did that made me understand it better.' (Ellen)

Her reflections indicated that these improved understandings were achieved through attunement to understanding mathematics from the perspective of teaching (e.g. comparing her own approaches with others' approaches and noting which approaches were simpler). Her improved understanding of mental mathematics thus reflected SCK, rather than a more CCK where the focus would be on completing problems effectively and efficiently for herself.

On the PCK side, all three PSTs were able to go beyond fluent and coherent articulations of the jump strategy in bringing together number line diagrams with alternative representations linked to the questions they described asking learners. There were also commonalities in the knowledge base related to a clear attentiveness and responsiveness to learner offers and needs. All three PSTs noted that their learners had struggled to distinguish between 'ten more' than a number and 'the next multiple of ten' after a given number. Tsolo articulated this with a specific example:

> '[*T*]hey only struggled with the multiples of 10, when you use, like, the multiples of 10 or jump by 10 for what is the next multiple of 10 from 56? So they jumped from 56 to 66. Like they struggled with the multiple.' (Tsolo)

Tsolo also noted that many of her learners initially attempted tasks using column methods and was specific about the kinds of errors that she saw learners making: 'most of them would get it wrong because they would say 71 – 32, and then they would say 2 minus one, and then get one'. By the time of the post-test, she noted that most had switched to sketching their own number line and could see that there was a greater incidence of correct answers amid this switch (borne out by the improvements set out in Table 1). Underpinning the increased facility with number line use, she acknowledged the class's strengths with place value decomposition that supported this progress: 'they know their place value very well – 10s, 100s and 1s, so it was easier using the number line in breaking down the second number'. Similarly, Ellen noted her learners' unfamiliarity with the part-part-whole bar diagram used in some of the assessment tasks (MSAP Teacher Guide, p. 37). Her responsiveness to this involved, like Hlonipha's example earlier, a small 'tweak' in the representational sequence that better connected with the representational repertoires that the class were familiar with:

> '[*T*]o help them, I suppose I, I just wrote it in a different way. So this one was, part was 10 and then the other part was blank and

the answer was 69. So what I just did was I wrote it as a sum, so 10 plus blank equals 69, and when I did that then all – well not all, but most of them used the number line to solve that and then they wrote in their answers in the blocks. They were a few that still didn't completely understand it, but most of them managed to understand it after I wrote it as a sum.' (Hlonipha)

With her translation of the part-part-whole representation into a number sentence form that her class had seen previously, she commented that most learners were able to proceed to work with the new representation. This kind of skill with connections is important in the face of South African evidence of weaknesses in bringing together diagrams, inscriptions and talk coherently (Askew et al. 2019) and its responsiveness to evidence of learner difficulties. Hlonipha similarly described her introduction of counting on or back in 10s with scenarios involving having some oranges and then buying, or giving away, 10 oranges at a time, further exemplifying her attention to a responsive sequence of tasks that would eventually lead to solving tasks like 36 + 12 using jump strategies.

Across these excerpts, we see SCK linked with KCS, with both combined and feeding into progressive and responsive KCT. Hlonipha's insertion of precursor tasks and Ellen's insertion of a number sentence matching a bar diagram representation exemplify decisions made about task presentations in teaching. While initially, all three PSTs stated that they used the MSAP starter activity sequence as presented in the resources, our probing indicated creative interim constructions of tasks that were intended to bridge experienced gaps for learners.

With the MSAP materials reflecting the 'curriculum' in use for the teaching tasks, we noted that all three PSTs showed careful and intentional attention to the content of the MSAP Teacher Guide. They were able to point to specific distinctions within the starter sequence – for example, Tsolo commented on addition being in focus across starters 1–4 before moving on to subtraction, and Hlonipha's aforementioned process of starting with the core strategy tasks in each starter and then looking back at the warm-ups to decide whether additional lead-ins were needed. Ellen's aforementioned noting of where she felt warm-up activities connected or did not connect with the focal strategy suggested a further step into critical commentary on task sequences and the connections between them. Across all the PSTs, this careful attention to curriculum appears to be underpinned by a strong CCK linked to the basic formulation of the structure of jump strategies. However, in Hlonipha's additions of tasks and Ellen's critique of the lack of connection between warm-up and main activities in some starters, there was also evidence of strengths in KCC underpinned by a well-developed SCK that allowed for well-formulated responsive task creations that mediated what they anticipated, or experienced, as challenges for learners.

Across all three PSTs, positive experiences of working with the MSAP materials were reported, with the materials described as clear, easy to use and useful in classrooms. Ellen commented on the ways in which the materials fed into the development of her own understanding of teaching in the offer of guidelines of how starters could be structured:

'[*The MSAP presented*] a very simple and straightforward way. And like the lesson steps that were given were extremely straightforward and it helped me to, to understand the lessons and … how the lessons are expected to proceed.' (Ellen)

Tsolo also mentioned that having the lesson plans offered useful support for her teaching. This suggests that carefully designed materials can be 'educative' for PSTs across several of Ball et al.'s (2008) MKT categories in simultaneous ways. This leads to our concluding discussion about what we can take away as both researchers and teacher educators from the reflections of these three high-gain PSTs.

Conclusion and discussion

At one level, having stories of success to share in the context of primary mathematics ITE in South Africa is important in its own right to counter the normalisation of the narrative that the problems in mathematics in the country are simply intractable. Sharing these stories, as outliers of a broader story of success is important to showing that change is possible across all kinds of classrooms.

Across the three teachers in our high gain group, there was evidence of relatively strong CCK linked to the structure of jump strategies and connected with SCK, a close knowledge of the content as presented in the MSAP 'curriculum', and a broader sense of the progressions that allow for responsive teaching that builds into the mental mathematics working presented in the materials. The PST responses also offer contrasts in how the relationship between fluency, calculation and equivalence tasks can be conceptualised. Our forward work includes the creation of some composite excerpts about different ways of seeing these relationships that can be shared and discussed with future ITE cohorts. One example of this would be to offer students statements to consider and discuss fictionalised 'key messages' linked with the MSAP materials:

- 'mental maths is all about not counting in ones';
- 'fluencies and strategies are two different parts of the MSAP lesson starters';
- 'fluencies are needed within children's work with the calculation strategy';
- 'calculations and equivalence tasks need to be taught separately'.

Opening up such discussions may offer us openings to strengthen the links between the different parts of the lesson starter structure in ways that support the mathematical connections between fluencies, calculations and equivalence within each strategy. Our aim here would be to highlight that fluencies, while important and necessary, are not ends in themselves, and that this idea of reified results as central to the learning of new processes, as Sfard (2008) emphasises, is one that is core to all mathematics learning. Our sense is that extending PSTs' knowledge with these insights – which perhaps can only happen in their reflections on initial teaching experiences linked to mental mathematics – will be crucial in extending the promising gains observed in the broader cohort.

Acknowledgements

The authors acknowledge the Wits and Rhodes numeracy chairs.

Competing interests

The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

Authors' contributions

H.V. and C.D.M. were involved in the collection and analysis of the data and the writing up of the article.

Funding information

This research received no specific grant from any funding agency in the public, commercial or not-for-profit sectors.

Data availability

The data that support the findings of this study are available from the corresponding author, C.D.M. upon reasonable request.

Disclaimer

The views and opinions expressed in this article are those of the authors and are the product of professional research. It does not necessarily reflect the official policy or position of any affiliated institution, funder, agency or that of the publisher. The authors are responsible for this article's results, findings and content.

References

Abdulhamid, L., 2017, 'Characterizing responsive primary mathematics teaching in the South African context', in M. Graven & H. Venkat (eds.), *Improving primary mathematics education, teaching and learning: Research for development in resource-constrained contexts*, pp. 199–216, Palgrave Macmillan, London.

Abdulhamid, L. & Venkat, H., 2018, 'Primary mathematics teachers' responses to students' offers: An "elaboration" framework', *The Journal of Mathematical Behavior* 51, 80–94. https://doi.org/10.1016/j.jmathb.2017.08.007

Askew, M., Venkat, H., Abdulhamid, L., Mathews, C., Morrison, S., Ramdhany, V. et al., 2019, 'Teaching for structure and generality: Assessing changes in teachers mediating primary mathematics', in *43rd conference of the international group for the psychology of mathematics education*, PME, Pretoria, South Africa, 07–12 July, 2019, pp. 41–48.

Askew, M., Graven, M. & Venkat, H., 2022, 'From what works to scaling up: Improving mental strategies in South African Grade 3 classes', in C. Fernández, S. Llinares, A. Gutiérrez & N. Planas (eds.), *Proceedings of the 45th conference of the international group for the psychology of mathematics education,* Universidad de Alicante, Alicante, Spain, 18–23 July, 2022, vol. 2, pp. 27–34.

Ball, D.L. & Bass, H., 2009, 'With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures', in paper presented at the 43rd jahrestagung der gesellschaft für didaktik der mathematik, Oldenburg, Germany, viewed n.d., from https://www.mathematik.uni-dortmund.de/ieem/BzMU/BzMU2009/BzMU2009-Inhalt-fuer-Homepage.htm.

Ball, D., Thames, M.H. & Phelps, G., 2008, 'Content knowledge for teaching: What makes it special?', *Journal of Teacher Education* 59(5), 389–407. https://doi.org/10.1177/0022487108324554

Baroody, A.J., Torbeyns, J. & Verschaffel, L., 2009, 'Young children's understanding and application of subtraction-related principles', *Mathematical Thinking and Learning* 11(1–2), 2–9. https://doi.org/10.1080/10986060802583873

Bowie, L., Venkat, H. & Askew, M., 2019, 'Pre-service primary teachers' mathematical content knowledge: An exploratory study', *African Journal of Research in Mathematics, Science and Technology Education* 23(3), 286–297. https://doi.org/10.1080/18117295.2019.1682777

Clements, D.H. & Sarama, J., 2020, *Learning and teaching early math: The learning trajectories approach*, Routledge, New York, NY.

Essien, A.A., Mathews, C., Tshesane, H., Weitz, M., Abdulhamid, L., Hoosen, T. et al., 2023, 'Developing design principles to enhance pre-service teachers' understanding of number structure and mathematical equivalence in early grade mathematics', *African Journal of Research in Mathematics, Science and Technology Education* 27(3), 239–254. https://doi.org/10.1080/18117295.2023.2216608

Graven, M. & Venkat, H., 2021, 'Piloting national diagnostic assessment for strategic calculation', *Mathematics Education Research Journal* 33(1), 23–42. https://doi.org/10.1007/s13394-019-00291-0

Hoadley, U., 2007, 'The reproduction of social class inequalities through mathematics pedagogies in South African primary schools', *Journal of Curriculum Studies* 39(6), 679–706. https://doi.org/10.1080/00220270701261169

Hopkins, S. & Bayliss, D., 2017, 'The prevalence and disadvantage of min-counting in seventh-grade: Problems with confidence as well as accuracy', *Mathematical Thinking and Learning* 19(1), 19–32. https://doi.org/10.1080/10986065.2017.1258613

Mathews, C., 2021, 'Teaching of division in Grade 3 and Grade 4: Disruptions to coherence', PHD report, University of the Witwatersrand, Johannesburg.

Porteus, K., 2023, 'Mapping design principles to instructional realities in early grade mathematics in South Africa: A framework for designing and evaluating learning and teaching support materials', *African Journal of Research in Mathematics, Science and Technology Education* 27(3), 350–366. https://doi.org/10.1080/18117295.2023.2255375

Porteus, K. & Mostert, I., 2022, 'Moving beyond counting in ones: A South African framework for number picture representations', in T. Mosiane, L. Mokotedi, D. Laubscher & S. Mahlab (eds.), *Proceedings of the 27th annual national congress of the association for mathematics education of South Africa*, AMESA Potchefstroom, SA, 27 June–01 July, 2022, pp. 177–198.

Sfard, A., 2008, *Thinking as communicating: Human development, the growth of discourses, and mathematizing*, Cambridge University Press, New York, NY.

Shalem, Y., Steinberg, C., Koornhof, H. & De Clercq, F., 2017, 'The what and how in scripted lesson plans: The case of the Gauteng primary language and mathematics strategy', *Journal of Education* 66(66), 13–36. https://doi.org/10.17159/2520-9868/i66a01

Spaull, N., Pampallis, I., Ardington, C., Sapire, I. & Isaac, P., 2022, 'Not Adding it Up: Grade 1 mathematics outcomes in the Eastern Cape & Limpopo', in H. Venkat & N. Roberts (eds.), *Early grade mathematics in South Africa*, Oxford University Press, Cape Town.

Stephens, M. & Ribeiro, A., 2012, 'Working towards algebra: The importance of relational thinking', *Relime* 15(3), 373–402.

Taylor, N., 2021, 'The dream of Sisyphus: Mathematics education in South Africa', *South African Journal of Childhood Education* 11(1), 1–12. https://doi.org/10.4102/sajce.v11i1.911

Venkat, H, Askew, M. & Graven, M., 2023, 'Working with early number algebraically: The Mental Starters assessment project', in P. Drijvers, C. Csapodi, H. Palmér, K. Gostonyi & E.Kónya (eds.), *Proceedings of the 13th Congress of the European Society for Research in Mathematics Education*, Budapest, 10–14 July, pp. 668–675.

Venkat, H., Graven, M., Morrison, S., Vale, P., Askew, M., Bowie, L. et al., 2021, 'Going national with the mental starters assessment project 2021: Model, outcomes and provincial foundation phase maths coordinators' experience', *Electronic Record of,* 13.

Wolcott, H.F., 1994, *Transforming qualitative data*, Sage, London.